Linking Period and Cohort Life-Expectancy Linear Increases in Gompertz Proportional Hazards Models

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Abstract

We derive simple formulae linking period and cohort life expectancy, as well as their rates of change, in Gompertz proportional hazards settings. They imply that a 2.5-year gain per decade in period life expectancy, observed by Oeppen and Vaupel (2002), corresponds to a 3.3-year gain on the respective cohort basis. The validity of the two theoretically derived approximate formulae, as well as the appropriateness of model assumptions, has been tested empirically on datasets from the Human Mortality Database.

1 Relationship

Suppose the mortality schedule of a population is described by

$$\mu(x, y) = e^{-\rho y}\, a_0\, e^{bx} \tag{1}$$

on a period basis, which corresponds to

$$\mu^C(x,y) = e^{-\rho y}\, a_0\, e^{(b-\rho)x} \tag{2}$$

on a cohort basis, i.e., by a Gompertz hazard that decreases with time y at a relative rate ρ at all ages x simultaneously. Denote by $e_0^P(y)$ and $e_0^C(y)$ the period and cohort life expectancy in y. Vaupel (1986) proved that (1) results in a linearly increasing $e_0^P(y)$ with an approximate slope ρ/b. We will prove that (2) implies linear increase in $e_0^C(y)$, too. Moreover, for $y \to \infty$

$$\frac{e_0^C(y)}{e_0^P(y)} \longrightarrow \frac{1}{1-\dot{e}_0^P(y)} \tag{3}$$

and

$$\dot{e}_0^C(y) \longrightarrow \frac{\dot{e}_0^P(y)}{1-\dot{e}_0^P(y)}, \tag{4}$$

where $\dot{e}_0^P(y) = de_0^P(y)/dy$ and $\dot{e}_0^C(y) = de_0^C(y)/dy$.

2 Proof of the relationship

The number of years lived in the next n by those who have survived to age x in year y is (Keyfitz and Caswell 2005: 30)

$$L(x,n,y) = \int_x^{x+n} \ell(v,y)dv, \tag{5}$$

where

$$\ell(x, y) = \exp\left\{-\int_0^x \mu(v, y)dv\right\} \tag{6}$$

is the survival function of an individual aged x at time y.

Substituting (1) and (6) in (5), assuming $\ell(0, y) = 1$, setting $x = 0$ and letting $n \to +\infty$, we get the following expression for period life expectancy at birth:

$$e_0^P(y) = \frac{1}{b} \exp\left\{\frac{a_0}{b}\, e^{-\rho y}\right\} E_1\left(\frac{a_0}{b}\, e^{-\rho y}\right) , \tag{7}$$

where

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt \tag{8}$$

is the exponential integral. A relationship, based on (2) and similar to (7), holds for $e_0^C(y)$, as well:

$$e_0^C(y) = \frac{1}{b-\rho} \exp\left\{\frac{a_0}{b-\rho}\, e^{-\rho y}\right\} E_1\left(\frac{a_0}{b-\rho}\, e^{-\rho y}\right) . \tag{9}$$

The limit of $e_0^C(y)/e_0^P(y)$ for $y \to \infty$ can be represented as

$$\lim_{y\to\infty} \frac{e_0^C(y)}{e_0^P(y)} = \frac{b}{b-\rho} \cdot \lim_{y\to\infty} \exp\left\{\frac{-a_0\rho}{b(b-\rho)}\, e^{-\rho y}\right\} \cdot \lim_{y\to\infty} \frac{E_1\left(\frac{a_0}{b}\, e^{-\rho y}\right)}{E_1\left(\frac{a_0}{b-\rho}\, e^{-\rho y}\right)} . \tag{10}$$

The first limit on the right-hand side of (10) is equal to 1. The arguments of both exponential integrals in the second limit tend to 0 for $y \to \infty$. This implies that both the numerator and the denominator will tend to $+\infty$ as

$$E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty} \frac{z^n}{n \cdot n!} \tag{11}$$

(see, for example, Abramowitz and Stegun 1964: 229), where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. Applying L'Hôpital's rule yields

$$\lim_{y\to\infty} \frac{E_1\left(\frac{a_0}{b}\, e^{-\rho y}\right)}{E_1\left(\frac{a_0}{b-\rho}\, e^{-\rho y}\right)} = \lim_{y\to\infty} \exp\left\{\frac{a_0\rho}{b(b-\rho)} e^{-\rho y}\right\} = 1 \tag{12}$$

As a result, for $y \to \infty$

$$\frac{e_0^C(y)}{e_0^P(y)} \longrightarrow \frac{b}{b-\rho} = \frac{1}{1-\frac{\rho}{b}} = \frac{1}{1-\dot{e}_0^P(y)}\,. \tag{13}$$

Finally, differentiating (9) with respect to y, we get

$$\dot{e}_0^C(y) \;=\; \frac{\rho}{b-\rho}\left(\exp\left\{-\frac{2\,a_0}{b-\rho}\, e^{-\rho y}\right\} + e^{-\rho y}\, a_0\, e_0^c(y)\right)\,, \tag{14}$$

which implies for $y \to \infty$

$$\dot{e}_0^C(y) \longrightarrow \frac{\rho}{b-\rho} = \frac{\rho/b}{1-\rho/b} = \frac{\dot{e}_0^P(y)}{1-\dot{e}_0^P(y)} \tag{15}$$

Q.E.D.

3 History and related results

Linearly increasing period life expectancy for the best-practice country from 1840 to present was detected by Oeppen and Vaupel (2002). One of the possible models that yields constant increase per time unit is (1), introduced by Vaupel (1986). Goldstein and Wachter (2006) discussed the latter as a special case of Linear Shift Models ($\rho/b = r$), under which the hazard rate at every age x in year y is given by the hazard rate at a younger age $x - ry$, $r > 0$, y years earlier: $\mu(x, y) = \mu(x - ry, 0)$. Wilmoth (2005) discussed the relationship between period and cohort mortality by comparing different mean life span measures in general model settings. Without specifying any functional interrelationship, Schoen and Canudas-Romo (2005) compare period and cohort life expectancy in the Gompertz proportional hazards framework with different shapes $\exp\{-f(y)\}$ of mortality progress over time y, including $f(y) = \exp\{-ry\}$, specified in (1). Canudas-Romo and Schoen (2005) study a Siler model with two different (constant) rates of mortality decline: one for infant and one for non-infant mortality. This model converges with time to 1 as levels of and improvements in infant mortality become negligibly small. Canudas-Romo and Schoen (2005) quantify the gaps and lags between period and cohort life expectancy over time in terms of the model parameters. In this paper, within the framework of Gompertz proportional hazards (1), we express in a simple functional form the relationship between period and cohort life expectancy as well as between their (constant) rates of change. The two relationships are *approximate* and *do not* depend on model parameters as it is the case in Canudas-Romo and Schoen (2005). Eq. (13) suggests that we can link period to cohort life expectancy by the following approximation

$$e_0^C(y) \approx \frac{e_0^P(y)}{1 - \dot{e}_0^P(y)}\,, \tag{16}$$

whereas (15) implies that

$$\dot{e}_0^C(y) \approx \frac{\dot{e}_0^P(y)}{1 - \dot{e}_0^P(y)} \tag{17}$$

Note that for $y \to \infty$

$$\exp\left\{\frac{-a_0\rho}{b(b-\rho)}e^{-\rho y}\right\} = 1 - \frac{a_0\rho}{b(b-\rho)}e^{-\rho y} + o(e^{-\rho y})$$

and, following (11),

$$\frac{E_1\left(\frac{a_0}{b}\, e^{-\rho y}\right)}{E_1\left(\frac{a_0}{b-\rho}\, e^{-\rho y}\right)} = 1 + \frac{E_1\left(\frac{a_0}{b}\, e^{-\rho y}\right) - E_1\left(\frac{a_0}{b-\rho}\, e^{-\rho y}\right)}{E_1\left(\frac{a_0}{b-\rho}\, e^{-\rho y}\right)} =$$

$$= 1 + \frac{\ln b - \ln(b-\rho) + \frac{a_0 \rho}{b(b-\rho)}\, e^{-\rho y} + o(e^{-\rho y})}{-\gamma - \ln a + \ln b + \rho y - \frac{a_0}{b-\rho}\, e^{-\rho y} + o(e^{-\rho y})},$$

which implies that for $a_0 = 0.00001$, $b = 0.14$, and $\rho = 0.027$, formula (16) overestimates $e_0^C(y)$ by 2.12%-2.39% for $y = 1, 2, \ldots, 100$. Relationship (16) is sensitive with respect to mortality progress. For instance, if $\rho = 0.01$, keeping all other parameters as before, we get an overestimate by 0.7%-0.8%. Lower a_0 and higher b increase the accuracy in (16).

Relationship (17), though underestimating cohort life-expectancy increase, is much more accurate that (16). Note that for $y \to \infty$

$$\dot{e}_0^C = \frac{\rho}{b-\rho}\left(1 + a_0\, e^{-\rho y}\left(-\frac{2}{b-\rho} - \gamma - \ln a + \ln b + \rho y\right) + o(e^{-\rho y})\right)$$

For $a_0 = 0.00001$, $b = 0.14$, and $\rho = 0.027$, the error in formula (17) is just between 4.05×10^{-6} and 8.47×10^{-5} for $y = 1, 2, \ldots, 100$.

Formulae (16) and (17) provide a simple approximate relationship between $e_0^P(y)$ and $e_0^C(y)$, as well as their rates of change. The two major assumptions under which this result has been derived are: 1) the same yearly mortality improvement at all ages, 2) the Gompertz assumption about the baseline hazard.

The Gompertz assumption about the adult force of mortality can be easily verified by looking at the logarithm of the unsmoothed mortality surface for Japan, the current life-expectancy leader (whose almost linear increase in life expectancy in the last decades is well pronounced). The mortality on a log-scale increases linearly with age in accordance with the exponentially increasing force of mortality. (Fig. 1)

Fig. 1 shows clearly the significance of age-invariant period effects, especially at older ages where the force of mortality is increasingly Gompertz-like. Only one significant cohort effect can be seen for the cohorts born during the Second World War.

The constant $\rho(x, y) = \rho(y)$ at all ages in a single year y has been verified in numerous studies including Kannisto (1994), Tuljapurkar et al. (2000) and Bongaarts and Feeney (2003), for the industrial countries in the second half of the 20th century. In this paper we assume

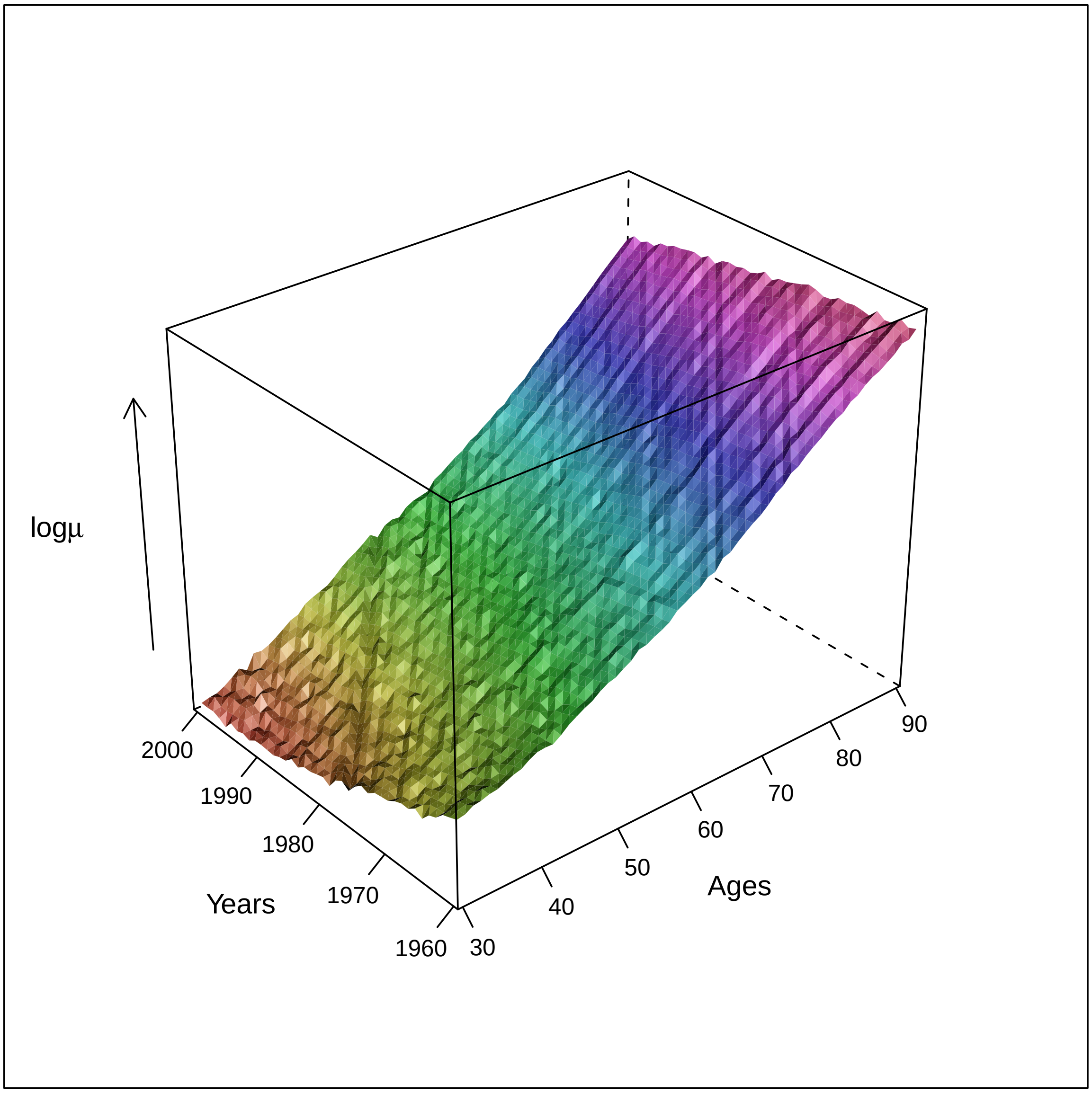

Figure 1: Logarithm of the observed force of mortality in Japan, ages 30-95, years 1960-2008

that ρ is constant not only over age x, but also with time y. Defining the observed rate of mortality change as

$$\rho(x, y+1) = -\log \frac{\mu(x, y+1)}{\mu(x, y)},$$

and plotting it in Fig. 2, using again the Japanese data, we do not observe any particular age or period dependent pattern. The rate of mortality change appears rather to be a cloud centered around the mean value 0.027. Life-expectancy increase in Japan has not obviously been one and the same each year: there have been years of decrease followed by years of fast increase (negative and very high ρ values, respectively). Nevertheless a constant ρ is a good approximation to the observed mortality processes. The orange points on Fig. 2 show the interval of one standard deviation distance from the mean. It encompasses 72% of the data. The purple points (two standard deviations distance from the mean) and the orange points constitute together 95% of the observed mortality changes.

Fig. 3 shows the density estimate for the observed mortality change in Japan. As the rates are tightly packed around the mean, they corroborate the notion of age and time-invariant mortality change.

Using Maximum Likelihood Estimation to obtain the optimal a, b and ρ parameters can help visualizing the error made by fitting a Gompertz model to Japanese mortality surface for ages 30-95m years 1960-2008. Fig. 4 shows the Q-Q plot for year 2008. If we compare the observed and expected distribution of ages at death, it can be seen that for ages under 50 the Makeham term would still have a significant effect and the Gompertz model underestimates the number of deaths. However, for older adult ages the Gompertz model predicts the age-specific number of deaths accurately even if we use a unique age and time constant mortality decline parameter for the whole mortality surface.

We derived (16) by using (7). The latter can be used to calculate life expectancy gains $G_{LE}(x, y)$ at every age x in each year y:

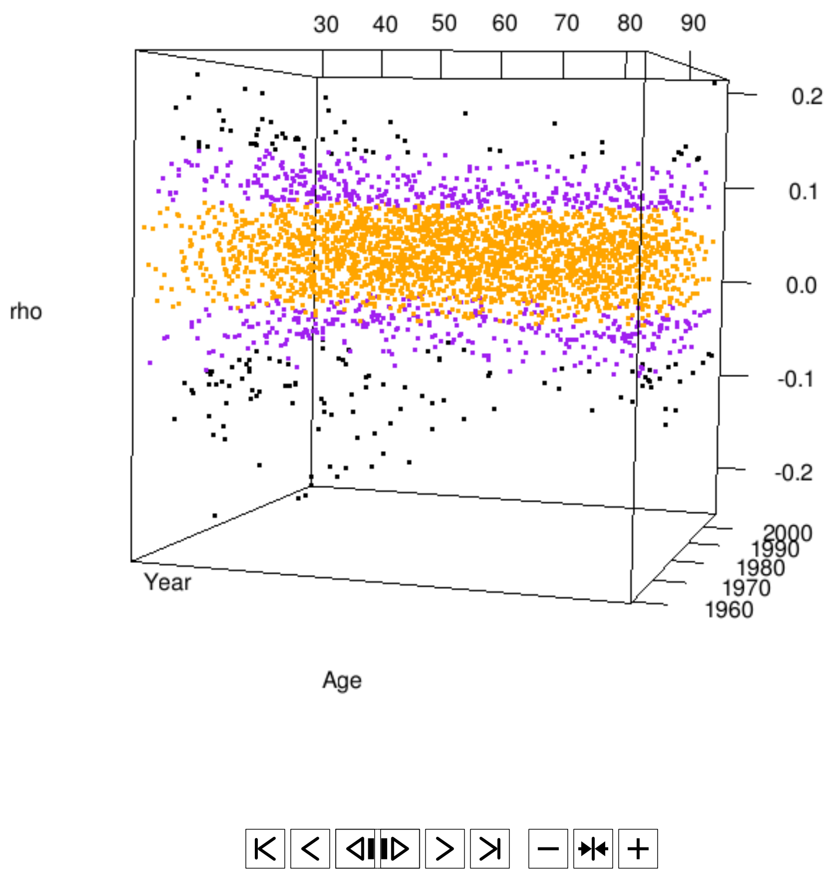

Figure 2: Observed rate of mortality change in Japan, ages 30-95, years 1960-2008. Orange band designates the ρ values that fall into the mean $\pm$ standard deviation. The ρ values in the purple band are in the mean $\pm$ two standard deviations interval.

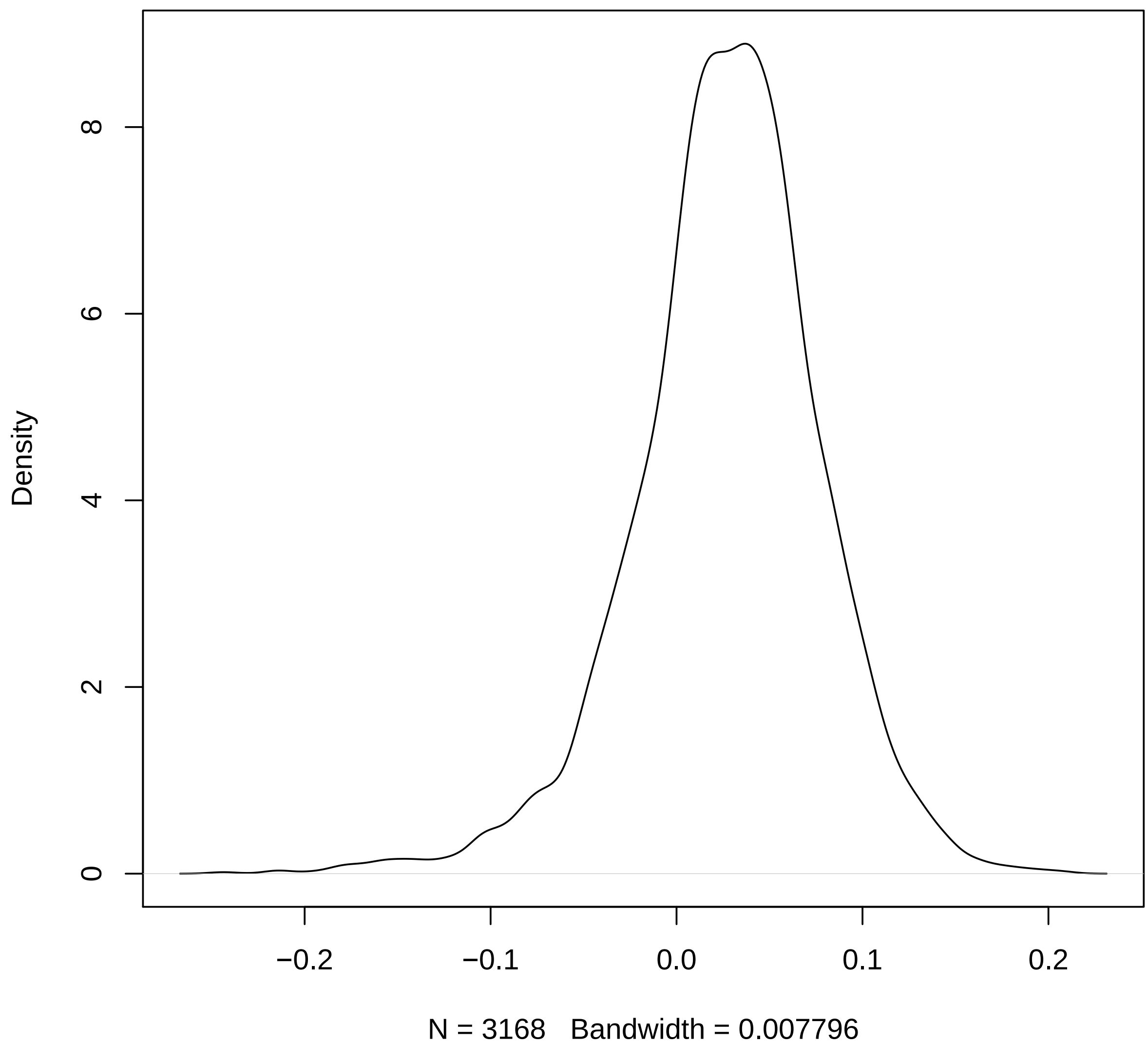

Figure 3: Density of observed rate of mortality change in Japan, ages 30-95, years 1960-2008 pooled together.

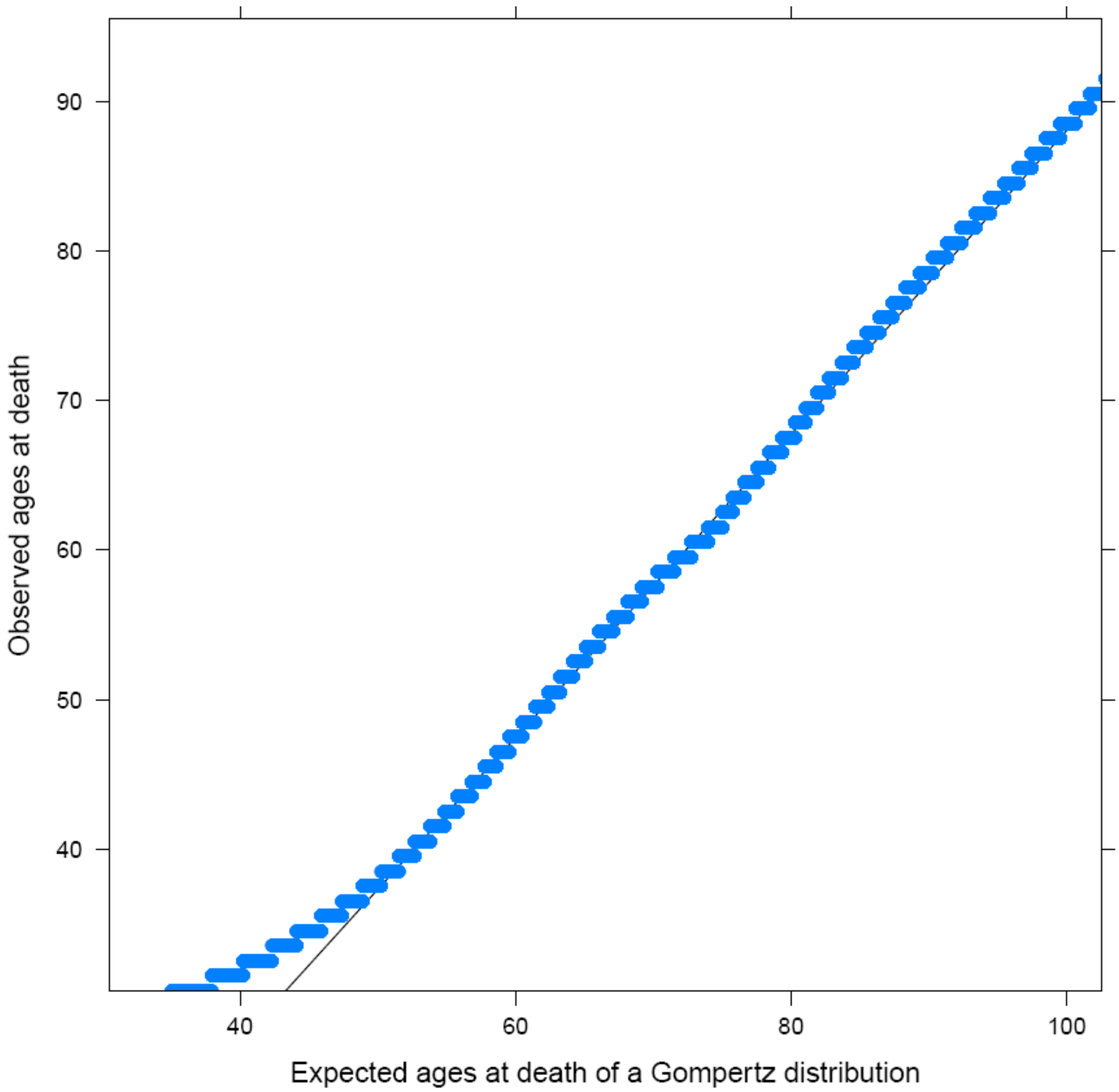

Figure 4: Q-Q plot of observed and expected ages at death for Japan ages 30-95 in 2008.

$$G_{LE}(x,y) = \frac{de_0(y)}{dy} = \frac{a_0\,\rho}{b}\,e^{-\rho y}(e^{bx}-1)\exp\left\{-\frac{a_0}{b}\,e^{-\rho y}(e^{bx}-1)\right\}. \tag{18}$$

Moreover, the age at which the maximal gains in $e_0^P(y)$ occur can be easily calculated as

$$x^*(y) = \frac{dG_{LE}(x,y)}{dx} = \frac{\ln(b+e^{-\rho y}a_0)+\rho y-\ln a_0}{b}. \tag{19}$$

Note that the age $x^*(y)$ at maximal gains in life expectancy shifts almost uniformly ($y \to \infty$) over time at a pace of

$$\frac{dx^*(y)}{dy} = \frac{\rho}{b}. \tag{20}$$

Thus the shift of $x^*(y)$ by periods (see Fig. 1) asymptotically ($y \to \infty$) equals the rate of change in life expectancy at birth.

4 Applications

A proportionally changing force of mortality leads to linearly shifting distribution of person-years gained by periods. In case of cohorts, this linear shift expands over time. Based on this relationship, cohort life expectancy at birth can be represented in terms of period life expectancy at birth and its increment. A similar representation applies to the rate of change in cohort life expectancy at birth.

Relationships (16) and (17) provide an approximate estimate of how period gains in life expectancy are transformed on a cohort basis. Consider individuals that were born in 2010 in a country with a period life expectancy at birth equal to 80 years. In this country, assume

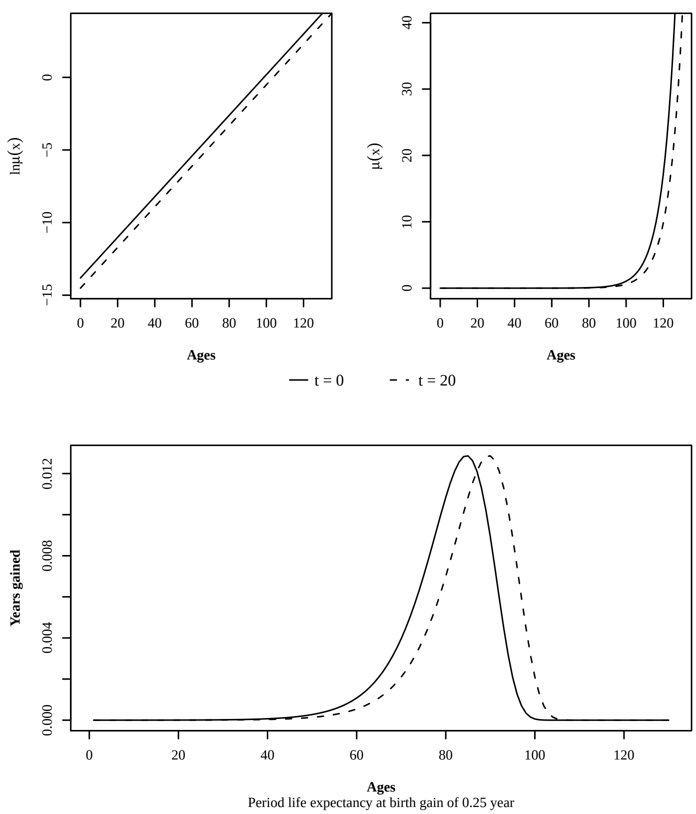

Figure 5: Change in the force of mortality and distribution by ages of years gained by constant period life expectancy at birth increase

that (i) there is a constant 2.5-year increase in life expectancy per decade (as in Oeppen and Vaupel 2002) and (ii) the force of mortality follows a Gompertz curve with constant yearly deceleration rate ρ. In a period perspective, this would mean that during their childhood and young adulthood these individuals gain additional six hours of life every day (Vaupel 2009: 352). Throughout the life course this gain shrinks because there are less remaining years, in which mortality can improve. In order to claim that individuals gain six hours a day in their infancy, one has to assume that they will experience the same probability of death, when they turn ten in 2020, as the one for a 10-year-old child in 2010. Similarly, when they turn 80 in 2090, they would have the same probability to die as a 80-year-old person in 2010. This assumption should hold on every day (or birthday, depending on the measurement units) of their life. However, within our assumptions, mortality improves in such a way that individuals aged 80 in 2090 would have the same probability to die as a 60-year-old person in 2010.

In a cohort view, when we account for the ever changing mortality regime, we can see that these life expectancy gains are not negligible. Individuals in this country will have a (cohort) life expectancy of 103.92 years. If we do not correct for overestimation, our result will be $\frac{80}{1-0.25} = 106.67$ years. Individuals born ten years later will have life expectancy, which will be higher by $\frac{2.5}{1-0.25} = 3.33$ years. Thus 2.5-year life expectancy gains per decade on a period basis correspond to approximately 3.3-year life expectancy gains per decade on a cohort basis.

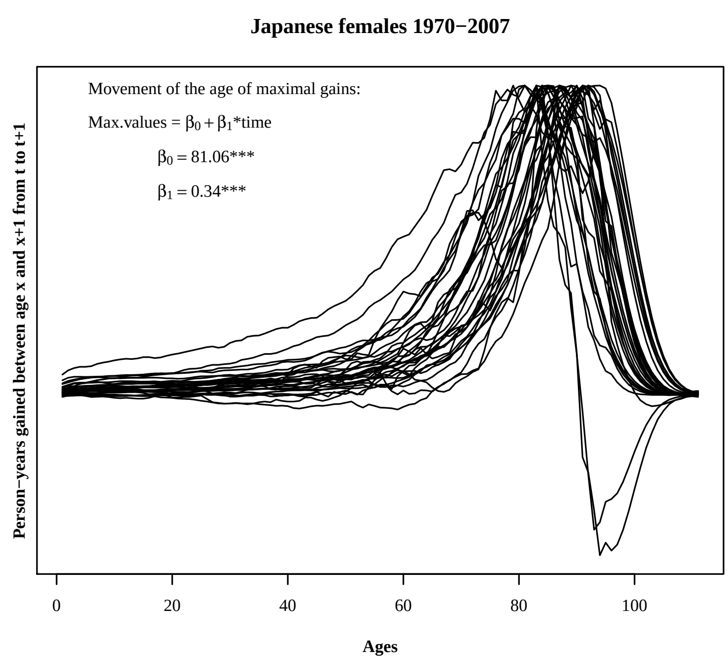

Figure 6: Distribution by ages of person-years gained

Fig. 6 shows an example of changes in person-years lived in an age interval between two consecutive periods. Using the data from the Human Mortality Database, we chose Japan females for demonstration. As life expectancy at birth augmentation over time is not constant even in Japan, we drew person-years lived changes from one period to another to the same scale and removed six years when life expectancy at birth was lower than in the previous year. The resulting age distribution of person-years gained and the movement of the curve resembles to the one predicted by the model (see Fig. 5).

Life expectancy gains result from the continuous linear shift to older ages in the distribution of person-years gained. Gompertz proportional hazards models stipulate reduction in the late middle-age and old-age mortality. Mortality of oldest-old, though, stays unmodified. However, as time advances, older and older ages benefit more from mortality reduction.

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